Keck's experiments were with a transmission line circuit. However, when the current becomes limited by the changing accelerator inductance rather than the external circuit, the eternal circuit has little effect on the accelerator current. Thus, the present computations are expected to accurately predict the speeds, particularly at speeds above 8 cm/μsec.

Patrick[6] has reported speeds in a coaxial accelerator from 20 to 40 cm/μsec. We have attempted to duplicate his experimental conditions, but the present models give speeds of 18–22 cm/μsec.

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A Parallel between Keplerian Integrals and Integrals of the Adjoint Equations

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A CLASSICAL development of the Keplerian vector constants, angular momentum and the perifocus vector (Hamilton's integral), is presented. A similar procedure is applied to the combined motion and adjoint equations to derive two additional vector constants, analogous in algebraic form to the Keplerian integrals, one which is apparently new. A full set of six independent integrals for the adjoint variables is obtained by adjoining to the two vector constants a known integral explicitly dependent upon time.

I. Introduction

There has been recent interest[1–2] in finding integrals of the system

$$\ddot{\bar{X}} = (F/m\Lambda)\bar{\lambda} - (\mu/R^3)\bar{X} \tag{1}$$

$$\ddot{\bar{\lambda}} = -(\mu/R^3)\{\bar{\lambda} - [3/R^2](\bar{\lambda}\cdot\bar{X})\bar{X}\} \tag{2}$$

where $\bar{X}$ and $\dot{\bar{X}}$ represent position and velocity vectors relative to an earth centered, orthogonal, nonrotating coordinate system with gravitational constant μ. The vector $\bar{\lambda}$ is adjoint to $\dot{\bar{X}}$ and the magnitudes of $\bar{\lambda}$, $\bar{X}$, and $\dot{\bar{X}}$ are denoted Λ, R, and V, respectively. Equations (1) and (2) govern the optimum (maximum payload) trajectory of a powered vehicle in an inverse square gravitational field. The vehicle's mass, m, is assumed to vary linearly with time on finite thrust arcs and to be constant on coast arcs. When the thrust magnitude, F, is allowed to become large or even unbounded the resulting optimal trajectories characteristically become some sequence of short thrust arcs ($F = F_{\text{MAX}} > 0$) and relatively long Keplerian coast arcs ($F = 0$). As a first step in finding integrals for Eqs. (1) and (2) one might consider only the coast arcs. In the special case where the thrust magnitude is unbounded and the thrust arcs are infinitesimal (impulses), the integrals for coast arcs may easily be extended to account for the impulses.

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II. Derivation of the Integrals

Unpowered flight ($F = 0$) is considered first. Conservation of angular momentum is shown by crossing Eq. (1) from the left by $\bar{X}$ and observing that

$$\bar{X} \times \ddot{\bar{X}} = (d/dt)(\bar{X} \times \dot{\bar{X}}) = (d/dt)(\bar{L}) = 0 \tag{3}$$

If one treats the combined Eqs. (1) and (2) similarly with $\bar{X}$ and $\bar{\lambda}$, then $\bar{\lambda} \times \ddot{\bar{X}} = -(\mu/R^3)\bar{X} \times \bar{\lambda}$ and $\bar{X} \times \ddot{\bar{\lambda}} = -(\mu/R^3)\bar{X} \times \bar{\lambda}$. Adding these equations yields

$$\bar{\lambda} \times \ddot{\bar{X}} + \bar{X} \times \ddot{\bar{\lambda}} = (d/dt)(\bar{\lambda} \times \dot{\bar{X}} + \bar{X} \times \dot{\bar{\lambda}}) = (d/dt)(\bar{L}^*) = 0 \tag{4}$$

Note that Eq. (4) holds also for powered arcs.

The similarity of the respective derivations of $\bar{L}$ and $\bar{L}^*$ and the algebraic forms motivates one to seek a corresponding relation to Hamilton's integral, $\bar{M}$. Continuing then, consider

$$(d/dt)(\dot{\bar{X}} \times \bar{L}) = -(\mu/R^3)\bar{X} \times \bar{L} = -(\mu/R^3)[(\bar{X}\cdot\dot{\bar{X}})\bar{X} - R^2\dot{\bar{X}}] \tag{5}$$

But

$$(d/dt)[(\mu/R)\bar{X}] = -(\mu/R^3)[(\bar{X}\cdot\dot{\bar{X}})\bar{X} - R^2\dot{\bar{X}}] \tag{6}$$

whence

$$(d/dt)[\dot{\bar{X}} \times \bar{L} - (\mu/R)\bar{X}] = (d/dt)(\bar{M}) = 0 \tag{7}$$

By analogy then

$$(d/dt)(\dot{\bar{X}} \times \bar{L}^* + \dot{\bar{\lambda}} \times \bar{L}) = -(\mu/R^3) \times [\dot{\bar{\lambda}}\cdot\bar{X} + \bar{\lambda}\cdot\dot{\bar{X}} - (3/R^2)(\bar{\lambda}\cdot\bar{X})(\bar{X}\cdot\dot{\bar{X}})]\bar{X} - (\mu/R^3)(\bar{\lambda}\cdot\bar{X})\dot{\bar{X}} - (\mu/R^3)(\bar{X}\cdot\dot{\bar{X}})\bar{\lambda} + (\mu/R)\dot{\bar{\lambda}} \tag{8}$$

The right-hand side of Eq. (8) is $(d/dt)\ [\mu/R\bar{\lambda} - (\mu/R^3)(\bar{\lambda}\cdot\bar{X})\bar{X}]$ from which one obtains

$$(d/dt)[\dot{\bar{X}} \times \bar{L}^* + \dot{\bar{\lambda}} \times \bar{L} - (\mu/R)\bar{\lambda} + (\mu/R^3)(\bar{\lambda}\cdot\bar{X})\bar{X}] = (d/dt)(\bar{M}^*) = 0 \tag{9}$$

The constant $\bar{M}^*$ has nearly the same interesting relation to $\bar{M}$ as does $\bar{L}^*$ to $\bar{L}$.

In the derivation of $\bar{M}^*$ with nonzero thrust terms included, Eq. (9) becomes

$$(d/dt)(\bar{M}^*) = (F/m\Lambda)\{\dot{\bar{\lambda}} \times (\bar{X} \times \bar{\lambda}) + \bar{\lambda} \times \bar{L}^*\} \tag{10}$$

As in Ref. 2, Eq. (10) may be integrated over infinitesimal thrust arcs. Thus at any time t, greater than the initial time t_o, in which p such thrust arcs have been traversed, each at times t_i $i = 1, \ldots, p$, Eq. (10) may be integrated

$$\int_{t_o}^{t} \frac{d}{dt}(\bar{M}^*) = \sum_{i=1}^{p} \int_{t_i^-}^{t_i^+} \frac{F}{m\Lambda}\{\dot{\bar{\lambda}} \times (\bar{X} \times \bar{\lambda}) + \bar{\lambda} \times \bar{L}^*\}$$

then

$$\bar{M}^* - \bar{M}_o^* = \sum_{i=1}^{p} \frac{1}{\Lambda_i}\{\dot{\bar{\lambda}} \times (\bar{X} \times \bar{\lambda}) + \bar{\lambda} \times \bar{L}^*\}_i \int_{t_i^-}^{t_i^+} \frac{F}{m}\,dt \tag{11}$$

This follows by the mean value theorem for integrals and the continuity of $\bar{\lambda}$, $\dot{\bar{\lambda}}$, and $\bar{X}$ over impulses.[3] The F/m integrals are merely the velocity increments at the times t_i.

III. Question of Independence

The six constants $\bar{L}$ and $\bar{M}$ do not supply a linearly independent set of integrals because the 6×6 matrix

$$T = \begin{bmatrix} \partial\bar{L}/\partial\bar{X} & \partial\bar{L}/\partial\dot{\bar{X}} \\ \partial\bar{M}/\partial\bar{X} & \partial\bar{M}/\partial\dot{\bar{X}} \end{bmatrix}$$

has rank 5. This may be shown directly by using elementary row operations on T. The integrals $\bar{L}^*$ and $\bar{M}^*$ as defined for unpowered flight by Eqs. (4) and (9) are linear in the adjoint variables $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ and therefore may be expressed

$$B\begin{bmatrix} \bar{\lambda} \\ \dot{\bar{\lambda}} \end{bmatrix} = \begin{bmatrix} \bar{L}^* \\ \bar{M}^* \end{bmatrix} \tag{12}$$

Upon calculating the matrix T one readily observes that $T = B$ from which it follows that $\bar{L}^*$ and $\bar{M}^*$ represent only five independent integrals. However, the relation given by Pines[1]

$$1/3\ \bar{\lambda}\cdot\dot{\bar{X}} + 2/3\ \dot{\bar{\lambda}}\cdot\bar{X} = b - H(t - t_o)$$

where H is the variational Hamiltonian ($F = 0$)

$$H = -(\mu/R^3)\bar{\lambda}\cdot\bar{X} - \dot{\bar{\lambda}}\cdot\dot{\bar{X}}$$

and b is an integration constant, may be adjoined to Eq. (12) to give a new linear system

$$\begin{bmatrix} B \\ 1/3\ \dot{\bar{X}}^T\ 2/3\ \bar{X}^T \end{bmatrix}\begin{bmatrix} \bar{\lambda} \\ \dot{\bar{\lambda}} \end{bmatrix} = \begin{bmatrix} \bar{L}^* \\ \bar{M}^* \\ b - H(t - t_o) \end{bmatrix} \tag{13}$$

which does have maximum rank for nonparabolic orbits. More specifically, if the 7×6 coefficient matrix of Eq. (13) is defined to be A, then it may be shown that the determinant of A^TA is

$$\mu^4R^2V^6[V^2 - 2(\mu/R)]^2$$

IV. Some Observations

For the case $F = 0$ the equations of variation of Eq. (1) defining the state transition matrices

$$\frac{\partial\bar{X}}{\partial\bar{X}_o}, \frac{\partial\bar{X}}{\partial\dot{\bar{X}}_o}, \frac{\partial\dot{\bar{X}}}{\partial\bar{X}_o}, \text{ and } \frac{\partial\dot{\bar{X}}}{\partial\dot{\bar{X}}_o}$$

satisfy Eq. (2). Thus the integrals $\bar{L}^*$ and $\bar{M}^*$ serve also as integrals for the state transition matrices. A completely analogous derivation gives the state transition analog of Eq. (13)

$$A\begin{bmatrix} \partial\bar{X}/\partial\bar{X}_o & \partial\bar{X}/\partial\dot{\bar{X}}_o \\ \partial\dot{\bar{X}}/\partial\bar{X}_o & \partial\dot{\bar{X}}/\partial\dot{\bar{X}}_o \end{bmatrix} = A_o$$

It is evident from Eq. (13) that if $H = 0$, then the adjoint variables are explicitly dependent upon the constants $\bar{L}^*$, $\bar{M}^*$, and b and the state $\bar{X}$ and $\dot{\bar{X}}$ only. Hence if $\bar{X}$ and $\dot{\bar{X}}$ are periodic and $H = 0$ then also are $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ with the same period. Moreover if $H \neq 0$ and $\bar{X}$ and $\dot{\bar{X}}$ are periodic then $\bar{\lambda}$ and $\dot{\bar{\lambda}}$ are not periodic.

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Spin Required to Limit Projectile Oscillations in a Finite Range

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1. Introduction

THE problem of spin stabilization is one of long standing. It was considered by Fowler (1920)[2] and the fundamental condition for complete stabilization was derived. The general equations of motion are very complex and it is difficult to extract from them simple criteria for the amount of spin necessary to restrict the yaw angle to some small amount. In a hyperballistic range, such as the one at RARDE, however the projectiles travel at very high speed over quite short distances and often at reduced pressures. In this case it is not unreasonable to simplify the problem by neglecting nonlinear effects, gravity, damping, and Magnus force. For instance the change in velocity due to gravity is about 10^{-6} projectile velocity and the changes due to damping and Magnus forces do not amount to more than a few per cent.

The simplified set of equations allow a very simple solution which shows that in this case the classical stability criterion is not sufficient to ensure that the oscillations are small. New criteria are therefore sought which will ensure this.

2. Theory

Consider an axisymmetric projectile whose moment of inertia about the axis of symmetry is I_1 and whose moment of inertia about an axis through the center of gravity normal to the axis of symmetry is I_2. We will denote the angle of yaw by α and the angle of pitch by β. If we let M be the aerodynamic pitching (or yawing as the projectile is axisymmetrical) moment per radian and let p_0 be the spin velocity then it is shown in Arnold and Maunder (1961)[1] that the perturbed motion obeys the equations

$$(I_2\ddot{\alpha} - M\alpha) - I_1p_0\dot{\beta} = 0 \tag{1a}$$

$$I_1p_0\dot{\alpha} + (I_2\ddot{\beta} - M\beta) = 0 \tag{1b}$$

when nonlinear effects, damping, gravity, and Magnus forces are neglected. Initially it will be assumed that the yaw angle is zero although the rate of yawing may be finite due to the flexing of the barrel as the projectile is emerging.

It is convenient to nondimensionalize Eq. (1) by setting

$$\tau = Vt/d,\ P = Md^2/I_2V^2, \text{ and } Q = I_1\,p_0d/I_2V \tag{2}$$

where V is the projectile velocity and d is the maximum projectile diameter.

The initial conditions can then be written as

$$\alpha = \beta = \beta' = 0 \text{ and } \alpha' = \Omega \text{ when } \tau = 0 \tag{3}$$

(For brevity Ω will be referred to as gun jump). The solution satisfying these conditions can easily be shown to be

$$\alpha = (2\Omega/\lambda)\cos(Q\tau/2)\sin(\lambda\tau/2) \tag{4a}$$

$$\beta = -(2\Omega/\lambda)\sin(Q\tau/2)\sin(\lambda\tau/2) \tag{4b}$$

where

$$\lambda = (Q^2 - 4P)^{1/2}$$

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